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/INFORMATION STRUCTURES
AND VIABLE PRICE SYSTEMS/

by

Chi-fu Huang*

January 1985; Revised - September 1985

Forthcoming in Journal of Mathematical Economics

WP #1574-84

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Abstract

A dynamic model of capital/financial markets is developed. A *surprise* is a stopping time that is not *foretellable*. We show that if agents' preferences exhibit a kind of time complementarity, then between ex-dividend dates a *viable price system* can make discrete changes only at surprises. Under the same preference condition, when the information in the economy can be modeled by a Brownian motion, a viable price system is an Ito process between ex-dividend dates. The martingale characterization of a viable price system originated by Harrison and Kreps (1979) is extended to our economy. This martingale result is independent of the time complementarity of preferences alluded to above.

1. Introduction

Much of the empirical work in financial economics and accounting concerns the response of capital/financial asset prices to information. The null hypothesis in this work is typically that the capital/financial markets are *efficient* in the sense that prices rapidly adjust to *new* information or *surprises*. The evidence to date has been largely interpreted as confirmation of this null hypothesis (e.g. Fama (1970) and Patell and Wolfson (1984)).¹ The construction of, and hence this interpretation of, the market efficiency tests has often been justified on intuitive grounds; for example, if the stock price of a takeover candidate is found to make *big* changes in response to the announcement of the takeover, stockholders are assumed to react to the information and the announcement takes them by surprise (see Jensen and Ruback (1983) for a survey of the empirical works on the market for corporate control).

To justify the intuitive interpretation of market efficiency test, we need a dynamic model of capital/financial markets. Is it true that prices can make discrete changes only at *new* information or *surprises*? To answer this question, *new* information or *surprises* will have to be mathematically defined and their linkage to prices formalized. Huang (1985) is the first paper in that direction. He considers a continuous time economy with a time span $[0, 1]$, where consumption is available only at two dates, 0 and 1. Using a long-lived asset as the numeraire, he shows that the sample path properties of a viable price system for long-lived assets are solely determined by the way information is revealed. A surprise is a stopping time that is not *predictable*. A price system can have jumps only at non-predictable stopping times, i.e. surprises.

The first question this paper addresses is to what extent can the results of Huang (1985) be generalized to accommodate consumption over time. Intuitively, if agents' consumption preferences change drastically over time, then the prices for long-lived assets can have jumps even at times of no surprise. On economic grounds, we expect agents' consumption preferences to exhibit some sort of time complementarity: a unit of consumption now and a unit of consumption an instant from now should be almost perfect substitutes. With this kind of time complementarity of consumption, it can be shown that between lump-sum (random) ex-dividend dates prices for long-lived assets can make discrete changes only at non-predictable stopping times. At lump-sum (random) ex-dividend dates, prices adjust downward by the amount of the dividends only when there are no surprises at that time.

For more than a decade financial theorists have been led to believe that Ito process representation of price processes is a good model for capital/financial markets not only because the tools involved are well-developed, but because the sample paths of an Ito process resemble the price

¹ Several anomalous and apparently persistent departures from the null must be explained in terms of test design to sustain this conclusion (see, for discussion, the June/September, 1978, issue of the Journal of Financial Economics).

chart of a typical common stock between (lump-sum) ex-dividend dates. That is, for most of the time, a typical common stock price moves up and down fast and in small steps.

Given that we have formulated a dynamic asset trading model, the second question addressed in this paper is under what conditions will a viable price system be an Ito process between ex-dividend dates. Sufficient conditions are that the information in the economy can be modeled by a Brownian motion and that agents' preferences exhibit the sort of time complementarity alluded to above.

Note that Huang (1985) has demonstrated the relationship between a Brownian motion information and Ito process representation of a viable price system. His result is numerated in a long-lived asset. The result reported here is in units of the natural numeraire, the single consumption commodity.

We also generalize the martingale result originated by Harrison and Kreps (1979) in the context of our economy. A price system for long-lived assets is viable if and only if it and its accumulated dividend process, both in units of a long-lived asset, follow a martingale under some probability measure which is uniformly absolutely continuous with respect to agents' commonly endowed probability. This generalization is valid in a very general economy without the sort of time complementarity needed for previously mentioned results.

The rest of this paper is organized as follows. Section 2 formulates the economy. Main results are in Sections 3 through 5. Section 6 contains discussions and concluding remarks.

2. The formulation

In this section we model a dynamic pure exchange economy under uncertainty with a time span $[0, 1]$. Taken as primitive is a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, P)$, where each $\omega \in \Omega$ denotes a complete description of the exogenous uncertain environment, where \mathcal{F} is the tribe of distinguishable events, where the *filtration*,² $\mathbf{F} = \{\mathcal{F}_t; t \in [0, 1]\}$, is the commonly endowed information structure of agents, and where P is the common probability measure on (Ω, \mathcal{F}) held by agents in the economy. We assume that \mathbf{F} satisfies the *usual* conditions:

1 *right-continuous*: $\forall t \in [0, 1], \mathcal{F}_t = \bigwedge_{s>t} \mathcal{F}_s, s \in [0, 1]$;

2 *complete*: $\forall t \in [0, 1], \mathcal{F}_t$ contains all the P -null sets.

We will also impose the conditions that agents learn the true state of the nature at time 1 and that now ($t = 0$) is certain. Mathematically, these entail that $\mathcal{F}_1 = \mathcal{F}$ and that \mathcal{F}_0 is almost trivial.

There is only one perishable consumption commodity in the economy which can be consumed at any $t \in [0, 1]$. We take the commodity space to be the space of integrable variation processes that have right continuous paths, denoted by V . (All the processes will be adapted to \mathbf{F} unless otherwise mentioned.) By definition, for each $v \in V$ we have:

² A *filtration* is an increasing family of subtribes of \mathcal{F} .

$$E \left(\int_{[0,1]} |dv(t)| \right) < \infty, \quad (2.1)$$

where $E(\cdot)$ denotes the expectation with respect to P . The intended interpretation is that $v(\omega, t)$ denotes the accumulated consumption from time 0 to time t in state ω . Let V_+ denote the space of nonnegative integrable increasing processes that have right continuous paths. It is clear that V_+ is a generating cone of V and we denote the order defined with respect to it by \geq .

It is assumed that the spot market for the consumption good is open all the time.

Remark 2.1. Since each $v \in V$ is right continuous and of bounded variation, it has left limits. Thus V is a space of RCLL (for right continuous with left limits) processes.

Remark 2.2. It is easily checked that if $v \in V$ then $E(v(t)) < \infty \forall t \in [0, 1]$.

Before proceeding, some definitions are in order. Let X be the space of RCLL processes such that

$$\text{ess sup}_{\omega, t} |x(\omega, t)| < \infty,$$

where the essential supremum is taken with respect to the probability measure P . We denote the nonnegative elements of X by X_+ , the positive cone of X . The order on X generated by X_+ is also denoted by \geq . Let X_{++} be a subset of X_+ satisfying the following: $\forall x \in X_{++}$ there exists $c \in \mathbb{R}_+$, $c \neq 0$ such that $P\{x(t) \geq c\} = 1 \forall t \in [0, 1]$.

Now consider the bilinear form $\psi : V \times X \mapsto \mathbb{R}$:

$$\psi(v, x) = E \left(\int_{[0,1]} x(t) dv(t) \right).$$

Proposition A.1 in Appendix I shows that ψ separates points, and places V and X in duality. Let τ be the strongest topology on V such that its topological dual is X , a Mackey topology; cf. Schaefer (1980, p.131).

Returning to economics, we shall assume that the economy is populated with agents from a set A , a generic element of which is denoted by α . Each agent $\alpha \in A$ is characterized by a utility function $U_\alpha : V \mapsto \mathbb{R}$ that is concave, τ -continuous, and strictly increasing, the last of which is in the sense that $U_\alpha(v + y) > U_\alpha(v) \forall y \in V_+$ and $y \neq 0$. We also assume that U_α has marginal utilities bounded below away from zero. Formally, let $\partial U_\alpha(v)$ denote the set of τ -continuous supergradients of U_α at v , which is nonempty by Theorem 14B of Holmes (1975). We assume that there exists $\gamma_\alpha \in X_{++}$ such that $\forall v \in V$

$$x \geq \gamma_\alpha \quad \forall x \in \partial U_\alpha(v).$$

Remark 2.3. An example of a τ -continuous, strictly increasing, and concave utility function having marginal utilities bounded below away from zero is given by $U : V \mapsto \mathbb{R}$, $U(v) = E(\int_{[0,1]} x(t)dv(t))$, where $x \in X_{++}$.

It is assumed that there are a finite number of long-lived securities traded in the economy, which are indexed by $j = 0, 1, 2, \dots, J$. Each long-lived security is represented by an element $D_j \in V_+$, where $D_j(\omega, t)$ denotes the accumulated dividends, in units of the consumption commodity, that a holder of a share of security j , from time 0 to time t , has received during that period in state ω . We shall assume that $D_0(t) = 0 \ \forall t \in [0, 1]$ and that $D_0(1)$ is bounded above and below away from zero. All the long-lived securities are traded ex-dividends. Thus without loss of generality, we also assume that $D_j(0) = 0 \ \forall j = 1, 2, \dots, J$.

An admissible price process for the single consumption commodity $\{\beta(t)\}$ is an RCLL process bounded above and below away from zero. A consumption plan is an element of V .

An admissible price system for traded long-lived securities is a $J + 1$ -vector of integrable semimartingales³ denoted by $S = \{S_j(t); j = 0, 1, \dots, J\}$. Since these securities are traded ex-dividends, $S_j(1) = 0 \ \forall j$.

Given admissible price systems for the single consumption commodity β and for long-lived securities S , an admissible trading strategy is an $J + 1$ -vector of predictable processes⁴ $\theta = \{\theta_j(t); j = 0, 1, \dots, J\}$ satisfying the following conditions:

$$1 \quad \text{ess sup}_{j, \omega, t} |\theta_j(\omega, t)| < \infty, \quad (2.2)$$

where as before the essential supremum is taken with respect to P ;

2 there exists $v \in V$, $v(0) = 0$ such that

$$\begin{aligned} \theta(t)^\top S(t) = \\ \theta(0)^\top S(0) + \int_0^t \theta(s)^\top dS(s) + \int_0^{t-} \beta(s)\theta(s)^\top dD(s) - \int_0^{t-} \beta(s)dv(s) \quad \forall t \in [0, 1] \quad a.s. \end{aligned} \quad (2.3)$$

and

$$\theta(1)^\top \Delta D(1) = \Delta v(1) \quad a.s., \quad (2.4)$$

where $^\top$ denotes inner product, and where $\Delta D(t)$ and $\Delta v(t)$ denote the jumps of D and v at t , respectively.

Equations (2.3) and (2.4) are the natural budget constraints, while (2.2) is a technical restriction. The budget constraints can best be understood when θ is left continuous. In that case, we

³For the definition of a semimartingale see Jacod (1979, p. 29). It is also shown there that for stochastic integrals to be well-defined and have nice properties it is necessary that the integrators be semimartingales; Jacod (1979, pp. 278-279).

⁴For the definition of a predictable process see Jacod (1979, p. 4).

interpret $\theta(t)$ to be the portfolio held from $t-$ to t before time t trading takes place. Thus $\theta(t)^\top S(t)$ is the value at time t of a portfolio carried from $t-$ into t , excluding dividends received at time t and before time t trading and consumption. That it is equal to the right-hand-side of (2.3) is clear.

Let $\Theta[\beta, S]$ denote the space of admissible trading strategies with respect to the pair of admissible price systems (β, S) . It is easily checked that $\Theta[\beta, S]$ is a linear space by the linearity of stochastic integrals.

In relations (2.3) and (2.4), the process v is said to be *financed* by θ , meaning obviously that if the strategy θ is followed, a consumption pattern represented by v can be *dynamically manufactured*. A consumption plan $v \in V$ is said to be *marketed* if $v - v(0)$ is financed by some $\theta \in \Theta[\beta, S]$. A price of v at time 0 is $\beta(0)v(0) + \theta(0)^\top S(0)$. Let M be the space of marketed consumption plans. By the fact that $\Theta[\beta, S]$ is a linear space, M is a linear space.

Remark 2.4. Since $\theta \in \Theta[\beta, S]$ is predictable and bounded, the stochastic integrals in (2.3) are well defined. See Dellacherie and Meyer (1982, Chapter VIII) for details.

3. Viability when the consumption good is the numeraire

Following Kreps (1981) we shall say that a pair of price systems (β, S) is *viable* (with respect to A) if it can be supported by some agent from the set A . Formally, (β, S) is viable if there exists $\alpha \in A$ and $v_\alpha \in V$ such that v_α is U_α -maximal in the set

$$\{v \in V : \beta(0)v(0) + \theta(0)^\top S(0) = 0 \text{ and } v \text{ is financed by } \theta \text{ for some } \theta \in \Theta[\beta, S]\}.$$

Note that the definition of *viability* is taken as primitive in this paper. The readers are referred to Kreps' paper for a host of related issues.

In this section we shall give characterizations of a viable pair of price systems, where the single consumption commodity is taken to be the numeraire. In this case, the consumption good price process is unit throughout, and we shall refer to a *viable price system* as a viable price system for the long-lived securities.

A necessary condition for a price system to be viable is that the single price law must hold for all the marketed consumption plans. In such event, there exists a linear functional $\pi : M \mapsto \mathbb{R}$ that gives prices at time zero of marketed consumption plans; that is, $\forall m \in M$, $\pi(m) = m(0) + \theta(0)^\top S(0)$, where $(m - m(0), \theta)$ are associated. We call π the *current price system* for M .

A τ -continuous linear functional $\psi : V \mapsto \mathbb{R}$ is said to be strictly positive if $\psi(v) > 0 \forall v \in V_+$ and $v \neq 0$.

Proposition 3.1. Suppose that S is viable. Then π has an extension to all of V denoted by ψ , which is strictly positive, τ -continuous, and can be represented as

$$\psi(v) = E \left(\int_{[0,1]} x(t) dv(t) \right) \quad \forall v \in V, \quad (3.1)$$

with $x \in X_{++}$.

Proof. This assertion is a direct consequence of the definition of viability. ■

The linear functional of (3.1) not only gives prices at time zero for marketed consumption plans, but also provides a way to represent a viable price system over time.

We shall assume throughout this section that S is viable.

Proposition 3.2. For every $j = 0, 1, \dots, J$, we have

$$\begin{aligned} S_j(t) &= \frac{E \left(\int_t^1 x(s) dD_j(s) \mid \mathcal{F}_t \right)}{x(t)} \\ &= \frac{E \left(\int_0^1 x(s) dD_j(s) \mid \mathcal{F}_t \right) - \int_0^t x(s) dD_j(s)}{x(t)} \quad \forall t \in [0, 1], \quad a.s., \end{aligned} \quad (3.2)$$

where RCLL versions of the conditional expectations are taken (this is possible because \mathbf{F} is assumed to be right continuous; cf. Meyer (1966, pp. 95–96)).

Proof. See Appendix II. ■

If we interpret the process $\{x(t)\}$ to be the *marginal utility* process, then Proposition 3.2 is just the famous relation: the price of a security at time t is equal to the conditional expectation at that time of the sum of products of its future dividends and the ratios of marginal utilities. This relation underlies much of the modern asset pricing theories; see, for example, Lucas (1978) and Rubinstein (1976).

Remark 3.1. From now on we shall always take an RCLL version of any conditional expectation that appears.

Several corollaries are immediate:

Corollary 3.1. The process $\{S_0(t) + D_0(t)\}$ is bounded above and below away from zero.

Proof. From Proposition 3.2 we know that $\forall t \in [0, 1)$

$$S_0(t) = \frac{E(x(1)D_0(1) \mid \mathcal{F}_t)}{x(t)} \quad a.s. \quad (3.3)$$

From Proposition 3.1 we know that x is bounded above and below away from zero. The assertion then follows from the assumption that $D_0(1)$ is bounded above and below away from zero. ■

Corollary 3.2. *The process $\{x(t)\}$ is a semimartingale.*

Proof. From (3.3) we can write

$$x(t) = \frac{E(x(1)D_0(1) \mid \mathcal{F}_t)}{S_0(t) + D_0(t)} \quad \forall t \in [0, 1] \quad a.s.$$

The numerator of the above expression is a martingale and the denominator is a semimartingale, which is bounded away from zero. A generalization of Ito's formula shows that $\{x(t)\}$ is a semimartingale; cf. Dellacherie and Meyer (1982, VIII.27). ■

Let $m \in M$ and let $\{S_m(t)\}$ denote its implicit ex-dividend price process. Given that S is a vector of semimartingales, a natural question is whether $\{S_m(t)\}$ is a semimartingale.

Corollary 3.3. *Let $m \in M$. Then $\{S_m(t)\}$ has a representation as (3.2) and is a semimartingale. Moreover, let $m - m(0)$ be financed by θ . Then*

$$S_m(t) = \theta(t)^\top S(t) + \theta(t)^\top \Delta D(t) - \Delta(m(t) - m(0)) \quad \forall t \in [0, 1] \quad a.s.$$

Proof. Applying the arguments found in the proof of Proposition 3.2, we can easily get

$$S_m(t) = \frac{E\left(\int_0^1 x(s) dm(s) \mid \mathcal{F}_t\right) - \int_0^t x(s) dm(s)}{x(t)} \quad \forall t \in [0, 1] \quad a.s.$$

The numerator of the above expression is a semimartingale, since it is the sum of a martingale and a bounded variation process. Corollary 3.2 shows that the denominator is a semimartingale too. The first assertion then follows from an application of the generalized Ito's formula.

The second assertion follows from the fact that S is viable. ■

When a price system is viable, we have stochastic integral representations for it. Moreover, the implicit ex-dividend price process for any marketed consumption plan and the *shadow price process* $\{x(t)\}$ are semimartingales.

4. Changing numeraire and martingales

We shall show in this section that if a numeraire is appropriately chosen, then the viability of a price system for long-lived securities necessitates that S , the price system, plus its accumulated dividend process, both in units of the numeraire, is a martingale under a probability measure on (Ω, \mathcal{F}) , which is uniformly absolutely continuous with respect to P . The converse is also true under some regularity condition.

Henceforth let (β, S) denote a pair of admissible price systems when the single consumption good is the numeraire, that is, $\beta(t) = 1 \forall t \in [0, 1]$. We will first show that S is viable if and only if the pair of price systems (β^*, S^*) is viable, where $\forall t \in [0, 1]$

$$\beta^*(t) = \frac{1}{S_0(t) + D_0(t)} \quad a.s.$$

and

$$S_j^*(t) = \beta^*(t) S_j(t) \quad \forall j = 0, 1, \dots, J \quad a.s. \quad (4.1)$$

This could be anticipated. In a Walrasian economy, changing numeraires ought to be economically neutral, since only relative prices are determined. This statement is not, however, always true in economies with infinitely many commodities. We will need to employ some special structures of our model.

We first show that (β^*, S^*) is a pair of admissible price systems if and only if (β, S) is.

Proposition 4.1. *(β^*, S^*) is a pair of admissible price systems if and only if (β, S) is a pair of admissible price systems.*

Proof. Suppose first that (β, S) is a pair of admissible price systems. The fact that β^* is bounded above and below away from zero and therefore admissible follows from Corollary 3.1. An application of the generalized Ito's formula shows that β^* is a semimartingale. The same formula also implies that S^* is a semimartingale. Finally, the fact that S^* is integrable follows from the above-proven fact that β^* is bounded above.

The proof for the converse is identical, so we omit it. ■

Let $\Theta[\beta^*, S^*]$ be the space of admissible trading strategies given that the pair of admissible price systems is (β^*, S^*) .

The following proposition shows that the two spaces $\Theta[\beta^*, S^*]$ and $\Theta[\beta, S]$ are equal. Its proof, being tedious and involving a lot of machinery in stochastic integration, is delegated to Appendix III.

Proposition 4.2. *$\theta \in \Theta[\beta, S]$ if and only if $\theta \in \Theta[\beta^*, S^*]$. Moreover, $v - v(0)$ is financed by θ given (β, S) if and only if it is financed by θ given (β^*, S^*) .*

Let M^* denote the space of marketed consumption plans given (β^*, S^*) . A corollary is immediate.

Corollary 4.1. $m \in M$ if and only if $m \in M^*$.

Proof. This is a direct consequence of the fact that $\Theta[\beta, S] = \Theta[\beta^*, S^*]$. ■

We shall henceforth use M to denote both M and M^* .

The following proposition shows that (β^*, S^*) is a pair of viable price systems if and only if (β, S) is one.

Proposition 4.3. (β^*, S^*) is a pair of viable price systems if and only if (β, S) is a pair of viable price systems. The linear functionals associated with two pairs of viable price systems, π and π^* , that give the prices at time zero of marketed consumption plans are linked by

$$\pi^*(m) = \frac{\pi(m)}{\pi(D_0)} \text{ and } \pi(m) = \frac{\pi^*(m)}{\beta^*(0)} \quad \forall m \in M.$$

Proof. Suppose first that (β, S) is viable. There must exist some $\alpha \in A$ and $v_\alpha \in V$ such that v_α is U_α -maximal in the set

$$\{v \in V : \beta(0)v(0) + \theta(0)^\top S(0) = 0 \text{ and } v - v(0) \text{ is financed by some } \theta \in \Theta[\beta, S]\}.$$

By Proposition 4.2 and the fact that $\beta(t) = 1 \ \forall t \in [0, 1]$, we know the above set is equal to the following set:

$$\{v \in V : \beta^*(0)v(0) + \theta(0)^\top S^*(0) = 0 \text{ and } v - v(0) \text{ is financed by some } \theta \in \Theta[\beta^*, S^*]\}.$$

Therefore, v_α is still U_α -maximal in the above set. By the definition of viability, the pair (β^*, S^*) is viable.

The proof for the converse is identical.

The proof for the rest of the assertion is easy. ■

It will be demonstrated in the sequel that S^* is closely related to martingales. Before proving this main result, we need a proposition and its corollary.

Putting

$$v^*(t) = \int_{[0,t]} \beta^*(s) dv(s) \quad \forall t \in [0, 1], v \in V,$$

and denoting the implicit ex-dividend price process of $m \in M$ given (β^*, S^*) by S_m^* , we have:

Proposition 4.4. *Suppose that the pair (β, S) is viable. Let $m \in M$. There exists $x^* \in X_{++}$ such that*

$$\begin{aligned} S_m^*(t) &= \frac{E\left(\int_t^1 x^*(s) dm^*(s) \mid \mathcal{F}_t\right)}{x^*(t)} \\ &= \frac{E\left(\int_0^1 x^*(s) dm^*(s) \mid \mathcal{F}_t\right) - \int_0^t x^*(s) dm^*(s)}{x^*(t)} \quad \forall t \in [0, 1], \quad a.s. \end{aligned} \quad (4.2)$$

Proof. Since $\pi^* = \beta^*(0)\pi$, π^* has an extension to all of V denoted by ψ^* . It is clear that ψ^* can be chosen to be $\beta^*(0)\psi$:

$$\psi^*(v) = \beta^*(0)E\left(\int_{[0,1]} x(t) dv(t)\right) \quad \forall v \in V, \quad (4.3)$$

where we recall that $x \in X_{++}$.

Now putting

$$x^*(t) = \frac{\beta^*(0)}{\beta^*(t)} x(t) \quad \forall t \in [0, 1], \quad (4.4)$$

x^* is strictly positive, bounded above and below away from zero.

We can rewrite (4.3) as

$$\psi^*(v) = E\left(\int_{[0,1]} x^*(t) dv^*(t)\right) \quad \forall v \in V.$$

Applying the same arguments to prove Proposition 3.2, it is straightforward to verify (4.2). ■

The interpretation of (4.2) is identical to that of (3.2), except that the numeraire here is not the consumption commodity. The following corollary is instrumental to the first main theorem of this section.

Corollary 4.2. *The process x^* defined in (4.4) is a martingale under P .*

Proof. We note that almost surely

$$\begin{aligned} D_0^*(t) &= 0 \quad \forall t \in [0, 1) \\ &= 1 \quad \text{if } t = 1 \end{aligned}$$

and

$$\begin{aligned} S_0^*(t) &= 1 \quad t \in [0, 1) \\ &= 0 \quad \text{if } t = 1 \end{aligned}$$

by construction. From Proposition 4.4 we have

$$\frac{E(x^*(1) \mid \mathcal{F}_t)}{x^*(t)} = S_0(t) = 1 \quad \forall t \in [0, 1) \quad a.s.$$

That is,

$$E(x^*(1) \mid \mathcal{F}_t) = x^*(t) \quad \forall t \in [0, 1] \quad a.s.$$

Naturally $E(x^*(1) \mid \mathcal{F}_1) = x^*(1)$ *a.s.* since x^* is adapted. These simply say that x^* is a martingale. ■

When appropriately numerated, the shadow price process is not only a semimartingale, it is a martingale!

Some definitions are in order.

A probability Q on (Ω, \mathcal{F}) is said to be *uniformly absolutely continuous* with respect to P if the Radon-Nikodym derivative dQ/dP is bounded above and below away from zero, denoted by $P \approx Q$. A *martingale measure* is a probability measure Q on (Ω, \mathcal{F}) with $Q \approx P$ such that $S^* + D^*$ is a martingale under Q .

Here is our first main theorem of this section:

Theorem 4.1. *Suppose that (β, S) is a pair of viable price systems. Then there exists a martingale measure Q . Moreover, $S_m^* + m^*$ is a martingale for all $m \in M$ under Q .*

Proof. It follows from Corollary 4.2 and Dellacherie and Meyer (1982, VI.57) that

$$\begin{aligned} S_m^*(t) &= \frac{E\left(\int_t^1 x^*(s) dm^*(s) \mid \mathcal{F}_t\right)}{x^*(t)} \\ &= \frac{E\left(\int_t^1 x^*(1) dm^*(s) \mid \mathcal{F}_t\right)}{x^*(t)} \\ &= \frac{E(x^*(1)(m^*(1) - m^*(t)) \mid \mathcal{F}_t)}{x^*(t)} \quad \forall t \in [0, 1] \quad P - a.s. \end{aligned} \tag{4.5}$$

Next note that $E(x^*(1)) = S_0(0) = 1$. Thus if we put

$$Q(B) = \int_B x^*(\omega, 1) P(d\omega) \quad \forall B \in \mathcal{F},$$

Q is a probability measure on (Ω, \mathcal{F}) uniformly absolutely continuous with respect to P . Therefore (4.5) can be written as

$$S_m^*(t) = E^*(m^*(1) - m^*(t) \mid \mathcal{F}_t) \quad \forall t \in [0, 1] \quad P - a.s.,$$

where $E^*(\cdot)$ denotes the expectation under Q ; cf. Gihman and Skorohod (1979, p.149). Since P and Q have the same null sets, the above expression holds with Q probability one. ■

When the pair of price systems (β, S) is viable, after an appropriate change of numeraire, the implicit price process for a marketed consumption plan plus its accumulated dividends will be a martingale under some probability uniformly absolutely continuous with respect to P .

Remark 4.1. Since P and Q have the same null sets, we shall from now on use *a.s.* to denote *almost surely* for both, unless otherwise mentioned.

Their proofs being straightforward, the following corollaries will simply be stated.

Corollary 4.3. S^* is a vector of supermartingales under Q .

Corollary 4.4. Let m_1, m_2 be such that $m_1^*(1) - m_1^*(t) = m_2^*(1) - m_2^*(t)$ a.s. for some $t \in [0, 1]$. Then

$$S_{m_1}^*(t) = S_{m_2}^*(t) \quad \text{a.s. and therefore } S_{m_1}(t) = S_{m_2}(t) \quad \text{a.s.}$$

At any time $t \in [0, 1]$, if the future accumulated dividends in units of $S_0 + D_0$ are equal for two marketed consumption plans, they must have the same price at that time.

Conversely, let (β^*, S^*) be a pair of admissible price systems with $S_0^*(t) + D_0^*(t) = 1 \forall t \in [0, 1]$ a.s. The following theorem shows that, under some regularity condition, for (β^*, S^*) to be viable, it suffices that there exists a martingale measure.

A proposition is first needed. Let M denote the space of marketed consumption plans given (β^*, S^*) .

Proposition 4.5. Suppose that there exists a martingale measure Q and that

$$E\left(\sup_{t \in [0, 1]} |S_j^*(t)|\right) < \infty \quad \forall j.$$

Then the single price law holds for all $m \in M^*$ and denoting the ex-dividend price of m at t by $S_m^*(t)$ we have

$$S_m^*(t) = E^* \left(\int_t^1 \beta^*(s) dm(s) \mid \mathcal{F}_t \right). \quad (4.6)$$

Proof. Let $\theta \in \Theta[\beta^*, S^*]$ be a trading strategy that finances m . Then an implicit price process for m is

$$S_m^*(t) = \theta(t)^\top S^*(t) + \beta^*(t) \theta(t)^\top \Delta D(t) - \beta^*(t) \Delta(m(t) - m(0)) \quad \forall t \in [0, 1] \quad \text{a.s.}$$

Equations (2.3) and (2.4) then imply that

$$S_m^*(t) = \int_t^1 \beta^*(s) dm^*(s) - \int_t^1 \theta(s)^\top d(S^*(s) + D^*(t)) \quad \forall t \in [0, 1], \quad \text{a.s.}$$

Given the hypotheses that (β^*, S^*) is viable and that $E(\sup_{t \in [0,1]} |S_j^*(t)|) < \infty$, it is easily verified that

$$E^* \left(\sup_{t \in [0,1]} |S_j^*(t) + D_j^*(t)| \right) < \infty \quad \forall j,$$

where $E^*(\cdot)$ is the expectation under Q . The Davis, Burkholder, and Gundy inequality (cf. Jacod (1979, 2.34)) then implies that

$$E^*([S^* + D^*, S^* + D^*]_1)^{\frac{1}{2}} < \infty, \quad (4.7)$$

where $\{[S^* + D^*, S^* + D^*]_t\}$ is the quadratic variation process of $\{S^*(t) + D^*(t)\}$. (For the definition of the quadratic variation process see Jacod (1979, 2.31).) Equation (4.7) as well as Jacod (1979, 2.48) ensures that

$$E^* \left(\int_0^t \theta(s)^\top d(S^*(s) + D^*(s)) \mid \mathcal{F}_t \right) = 0.$$

This in turn implies

$$S_m^*(t) = E^* \left(\int_t^1 \beta^*(s) dm(s) \mid \mathcal{F}_t \right),$$

which is independent of θ . Thus the single price law holds and (4.6) is valid. ■

Here is our second main result of this section:

Theorem 4.2. *Suppose that the conditions in Proposition 4.5 are valid. Then (β^*, S^*) is viable.*

Proof. Putting

$$\xi(t) = E \left(\frac{dQ}{dP} \mid \mathcal{F}_t \right) \quad \forall t \in [0, 1] \quad a.s.,$$

we have

$$E^* \left(\int_{[0,1]} \beta^*(t) dm(t) \right) = E \left(\int_{[0,1]} \xi(t) \beta^*(t) dm(t) \right) \quad \forall m \in M^*,$$

where we have again used Dellacherie and Meyer (1982, VI.57). From Proposition 4.5 we know that the two sets

$$\{v \in V : \theta(0)^\top S^*(0) + \beta^*(0)v(0) = 0, \text{ where } v \text{ is financed by some } \theta \in \Theta[\beta^*, S^*]\}$$

and

$$\{m \in M^* : E^* \left(\int_{[0,1]} \xi(t) \beta^*(t) dm(t) \right) = 0\}$$

are identical. Now define

$$U_\alpha(v) = E \left(\int_{[0,1]} \xi(t) \beta^*(t) dv(t) \right) \quad \forall v \in V.$$

It is quickly checked that $\alpha \in A$, since $\beta^* \xi$ is bounded above and below away from zero. It then follows from the separating hyperplane theorem that the pair of price systems (β^*, S^*) can be supported by U_α and thus is viable.

■

Whenever the regularity condition posited in Proposition 4.5 is satisfied, the existence of a martingale measure is equivalent to the viability of (β^*, S^*) and therefore the viability of (β, S) .

The combination of Theorems 4.1 and 4.2 is a generalization of the martingale result of Harrison and Kreps (1979). They consider an economy where consumption is available only at two dates. Here, consumption is allowed at any time in $[0, 1]$ and agents can consume at rates as well as in gulps.

Once we have the martingale characterization of viable price processes $S_m^*, m \in M$, it is easy to see that the sample path properties of S_m^* depend both upon the way information is revealed over time and upon the sample path properties of m^* . All the results of Huang (1985) can be generalized in a straightforward manner to show the interrelationship among S_m^* , m^* , and F . We leave this exercise to interested readers.

The sample path properties of S_m^* are not the most interesting objects in our economy, however. In the economy considered by Huang (1985), consumption is available only at two dates, time 0 and time 1. Since the consumption good is not available all the time, it is thus not unnatural to take a long-lived security as the numeraire. In the economy we are currently considering, the single consumption good is available all the time. The natural objects of interest ought to be the sample path properties of S_m . In particular, given the popularity of Ito process formulations of financial markets, it is important to see under what conditions S_m is an Ito process. Examining (3.2) we can see that the sample path properties of the price process of a marketed consumption plan depend not only on its accumulated dividend process and the information structure as captured by the conditional expectation, but also on the sample path properties of x . This is natural, since embedded in the sample path properties of x is a sense of time complementarity of consumption exhibited by preferences from the the set A . Given that viability is defined with respect to A , nothing can be said about x except that it is bounded. To say something interesting about x and therefore about $S_m, m \in M$, we have to consider a subset of A . This is the subject to which we now turn.

5. An alternative formulation

Let H denote a subspace of X such that each $h \in H$ is a continuous process.⁵ Let $H_{++} = H \cap X_{++}$. Define a bilinear form $\phi : V \times H \mapsto \mathfrak{R}$ by

$$\phi(v, h) = E \left(\int_{[0,1]} h(t) dv(t) \right).$$

Now identify equivalence classes of elements of V such that v_1, v_2 belong to an equivalence class if and only if

$$\phi(v_1 - v_2, h) = E \left(\int_{[0,1]} h(t) d(v_1(t) - v_2(t)) \right) = 0.$$

Denote the space of equivalence classes of V by \mathcal{V} . Then ϕ places \mathcal{V} and H in duality. Let τ^* be the strongest topology on \mathcal{V} such that its topological dual is H , a Mackey topology; cf. Schaefer (1980, p.131).

Let $\mathcal{A} \subset \mathcal{A}$ be such that each $U_\alpha, \alpha \in \mathcal{A}$ is τ^* -continuous. Note that the preferences of agents from the set \mathcal{A} exhibit the following sort of time complementarity of consumption: a unit of consumption now is almost a perfect substitute for a unit consumption an instant from now. This is so because the *shadow price* process is continuous.

Remark 5.1. Note that time-additive utility functions are not τ^* -continuous, since consumption at adjacent dates are perfect nonsubstitutes; cf. Huang and Kreps (1983).

Suppose from now on that S is viable with respect to \mathcal{A} . Still denote by M the space of marketed consumption plans. The following proposition can be viewed as a corollary of Proposition 3.2.

Proposition 5.1. *Let $m \in M$. Then $\{S_m(t)\}$ is a semimartingale and can be represented as*

$$S_m(t) = \frac{E \left(\int_0^1 h(s) dm(s) \mid \mathcal{F}_t \right) - \int_0^t h(s) dm(s)}{h(t)} \quad \forall t \in [0, 1] \quad a.s.,$$

where $h \in H_{++}$.

Now we are ready to examine the interrelationship among the accumulated dividend process of a marketed consumption plan, the sample path properties of its price process, and the way information is revealed. Some definitions are in order.

⁵ A process h is continuous if for almost every $\omega \in \Omega$, $h(\omega, \cdot)$ is a continuous function of time.

A stopping time is a map $T : \Omega \mapsto \mathbb{R} \cup \{\infty\}$ such that $\{T \leq t\} \in \mathcal{F}_t$ for all $t \in \mathbb{R}$, where we have used the convention that $\mathcal{F}_t = \mathcal{F} \forall t \geq 1$. A stopping time is said to be predictable if there exists a sequence of stopping times $(T_n)_{n=1}^\infty$ such that $T_n \leq T$ a.s. and $T_n < T$, $\lim T_n = T$ on $\{T > 0\}$. This sequence of stopping times is said to *foretell* T . (For detailed discussion of predictable stopping times see Chapter IV of Dellacherie and Meyer (1978).)

The idea of a stopping time is that of the first time some given random phenomenon occurs. To be more precise, any stopping time T can be interpreted as the *debut* of the set $\{(t, \omega) : t > T(\omega)\}$; cf. Dellacherie (1978, IV.51). “The existence of a foretelling sequence means that this phenomenon cannot take us by surprise: we are forewarned by a succession of precursory signs, of the exact time the phenomenon will occur;” cf. Dellacherie and Meyer (1978, IV.72).

Thus an event whose first occurrence is a predictable stopping time is not a *surprise*. On the other hand, an event whose first occurrence is not a predictable stopping time may take us by surprise with a strictly positive probability.⁶

We make the following regularity assumption throughout the rest of this section.

Assumption 5.1. *The information structure \mathbf{F} is quasi-left continuous. That is, let T be a predictable stopping time and let $\{T_n\}_{n=1}^\infty$ be any increasing sequence of stopping times foretelling T . Then*

$$\mathcal{F}_T = \bigvee_{n=1}^{\infty} \mathcal{F}_{T_n}.$$

An information structure satisfying the above assumption is also termed “having no time of discontinuity” by probability theorists. It is known, for example, that an information structure generated by a Markov process is quasi-left continuous, provided that the semigroup is nice; cf. Meyer (1963, Theorem 10).

The following proposition shows that the price process of a marketed consumption plan can be discontinuous only at either surprises or lump-sum ex-dividend dates.

Proposition 5.2. *Suppose that $m \in M$. Let $T : \Omega \mapsto [0, 1]$ be a stopping time. If $S_m(T) \neq S_m(T-)$ on a set of strictly positive probability, then either T is not predictable or $m(T) \neq m(T-)$ on a set of strictly positive probability.*

Proof. Suppose this is not true. From Proposition 5.1 and Meyer (1966, VI.T14) we know that $S_m(T-) = S_m(T)$ a.s., a clear contradiction. ■

⁶More precisely, a *totally inaccessible* stopping time is a surprise with probability one. See Dellacherie and Meyer (1978, IV.78) for details.

Between (random) lump-sum ex-dividend dates, the price process of a marketed consumption plan can have discrete changes only at non-predictable stopping times, or equivalently, at events that are not foretellable.

Proposition 5.3. *Suppose that $T : \Omega \mapsto \mathfrak{R}$ is a predictable stopping time. Then*

$$S_m(T) - S_m(T-) = - (m(T) - m(T-)) \quad a.s. \quad \forall m \in M.$$

Proof. When T is predictable, we know

$$E \left(\int_0^1 h(s) dm(s) \mid \mathcal{F}_T \right) = E \left(\int_0^1 h(s) dm(s) \mid \mathcal{F}_{T-} \right) \quad a.s.$$

The assertion then follows immediately from Proposition 5.1. ■

At lump-sum ex-dividend dates, which can be random, the price process would drop by the amount of the dividends when there are no surprises.

Before proceeding, a definition is in order.

Definition 5.1. An information structure is said to be continuous if all the stopping times are predictable.

Remark 5.2. For an equivalent definition and extensive discussions of a continuous information structure see Huang (1985).

The following is a direct consequence of the above definition and Proposition 5.3.

Corollary 5.1. *Suppose that \mathbf{F} is continuous. Then for any stopping time T*

$$S_m(T) - S_m(T-) = - (m(T) - m(T-)) \quad a.s. \quad \forall m \in M.$$

When the information structure is continuous, the ex-dividend date behavior of a price process is dictated by its dividend process. On the other hand, when a dividend process is continuous, the behavior of its price process is solely determined by the way information is revealed. Thus it can be the case that dividends and information work against each other and there won't be price changes at lump-sum ex-dividend dates.

For stopping times T_1 and T_2 , the set

$$[T_1, T_2) = \{(\omega, t) \in \Omega \times [0, 1] : T_1(\omega) \leq t < T_2(\omega)\}$$

is called a *stochastic interval*; cf. Chung and Williams (1983, p.29).

The following proposition shows that when the information structure is generated by a Brownian motion the price process of a marketed consumption plan is an Ito process in any stochastic interval within which the dividend process is continuous.

Proposition 5.4. *Suppose that \mathbf{F} is a Brownian motion filtration, that T_1, T_2 are two stopping times with $0 \leq T_1 < T_2 \leq 1$, and that $m \in M$ is a continuous process on $[T_1, T_2]$. Then $\{S_m(\omega, t); (\omega, t) \in [T_1, T_2]\}$ is an Ito process.*

Proof. First we note that a Brownian filtration is a continuous information structure; cf. Proposition 5.1.1 of Huang (1985). From the hypotheses and Corollary 5.1 we then know that $\{S_m(\omega, t); (\omega, t) \in [T_1, T_2]\}$ is a continuous semimartingale. By the defining properties of a semimartingale, $\{S_m(\omega, t); (\omega, t) \in [T_1, T_2]\}$ can be written as the sum of a local martingale and a process of bounded variation. It follows from Dellacherie and Meyer (1982, VIII.62) that the local martingale part can be represented as an Ito integral, therefore continuous. The bounded variation part is thus also continuous. Hence the price process is representable as the sum of an Ito integral and a continuous bounded variation process, and by definition is an Ito process. ■

Huang (1985) has shown in an economy where consumption is available only at two dates that a price process can be represented as an Ito integral when the information structure is a Brownian filtration. The above proposition is a generalization. When agents' shadow price process for consumption is a continuous process and when the information structure is generated by a Brownian motion, the price process between lump-sum ex-dividend dates will be an Ito process. Since there can be at most a countable number of (random) lump-sum ex-dividend dates, for most of the times, the price process of a marketed consumption plan is an Ito process.

The results of this section depend largely upon the fact that agents' preferences are τ^* -continuous. This assumption can be defended on economic grounds. The τ^* -continuity of preferences means that consumption at adjacent dates are almost perfect substitutes, which captures an intuitively appealing notion of time complementarity of consumption preferences over time. For more discussions on the issues of time complementarity of consumption see Huang and Kreps (1983).

6. Concluding remarks

It should be clear that results of Sections 3 and 4 do not depend upon the hypothesis that the time set is continuous. When the time set is any countable subset of $[0, 1]$, we simply change the integration signs to be summation signs. No interesting sample path properties of a viable price system for long-lived securities are available.

All of our analyses can be generalized in a straightforward manner to accommodate heterogeneous beliefs, as long as agents' endowed probability measures are uniformly absolutely continuous with respect to one another.

The results in this paper are strongly colored by the assumption that the consumption space of agents is the whole space. That is, negative consumption is allowed. This assumption is crucial

in establishing that the current price system for marketed consumption plans is representable as a τ -continuous linear function when a pair of price systems (β, S) is viable with respect to A and the results thereafter. The fact that relaxing this assumption may significantly change our results is evident from Jones (1984). What further restriction on A is needed for our analysis to go through when agents' consumption set is the positive orthant remains an open question. It seems that some sort of *properness* of preferences may be needed; cf. Mas-Colell (1983).

Appendix I

Proposition A.1. *The bilinear form $\psi : V \times X \mapsto \mathfrak{R}$:*

$$\psi(v, x) = E \left(\int_{[0,1]} x(t) dv(t) \right)$$

separates points.

Proof. Let $x_1, x_2 \in X$ and $x_1 \neq x_2$. Since x_1 and x_2 are RCLL processes, we know that there must exist $t \in [0, 1]$ and $B \in \mathcal{F}_t$ with $P(B) > 0$ such that

$$x_1(\omega) \neq x_2(\omega) \quad \forall \omega \in B.$$

Without loss of generality we assume that $x_1(\omega) > x_2(\omega)$ on B . Define $v : \Omega \times [0, 1] \mapsto \mathfrak{R}$ by

$$v(\omega, s) = 1_{B \times [t, 1]}(\omega, s).$$

Then

$$\begin{aligned} \psi(v, x_1) - \psi(v, x_2) &= E \left(\int_{[0,1]} (x_1(s) - x_2(s)) dv(s) \right) \\ &= E((x_1(t) - x_2(t)) 1_B) > 0. \end{aligned}$$

Thus V separates points in X .

Next let $v \in V$, $v \neq 0$. Then there must exist $t \in [0, 1]$ and $B_1 \in \mathcal{F}_t$ with $P(B_1) > 0$ such that $v(\omega, t) \neq 0 \quad \forall \omega \in B_1$. Without loss of generality we assume that $v(t) - v(t-) > 0$ on B_1 . Take the following cases.

Case 1. Suppose that $v(\omega, s) = v(\omega, t)$ for all $s \in [t, 1]$ and for almost every $\omega \in B_1$. Define $x : \Omega \times [0, 1] \mapsto \mathfrak{R}$ by

$$x(\omega, s) = 1_{B_1 \times [t, 1]}(\omega, s).$$

Then $x \in X$ and

$$\psi(v, x) = E(1_{B_1}(v(t) - v(t-))) > 0.$$

Case 2. Suppose that $|v(s) - v(t)| \geq v(t) - v(t-) \quad \forall s \in [t, 1]$ for almost every $\omega \in B_1$. This is impossible by the right continuity of v .

Case 3. There exists $\epsilon > 0$ such that the optional random variable $T_\epsilon : \Omega \mapsto [0, \infty]$ defined by

$$T_\epsilon = \inf\{s \in [t, 1] : \epsilon \leq |v(s) - v(t)| < v(t) - v(t-)\},$$

where we have used the convention that $T_\epsilon = \infty$ if the infimum does not exist, is finite on some $B_2 \in \mathcal{F}$ with $B_2 \subset B_1$ and $P(B_2) > 0$. Define $x : \Omega \times [0, 1] \mapsto \mathfrak{R}$ by

$$x(\omega, s) = 1_{B_2} 1_{[t, T_\epsilon]}(\omega, s).$$

Since $[t, T_\epsilon]$ is a stochastic interval, x is adapted, and by construction it is RCLL; cf. Chung and Williams (1983, Chapter 2). That is, $x \in X$. Then

$$\begin{aligned}\psi(v, x) &= E \left(\int_{[0,1]} x(s) dv(s) \right) \\ &= E(1_{B_2}(v(T_\epsilon) - v(t-)) > 0).\end{aligned}$$

Combining the above three cases, we have shown that X separates points in V . ■

Appendix II: Proof for Proposition 3.2.

Proof. Consider long-lived security j . At time zero, (3.2) is certainly true, since $x(0) = 1$ by the fact that the consumption commodity is taken to be the numeraire. Thus if (3.2) is not true, there must exist $t \in (0, 1]$ and $B \in \mathcal{F}_t$ with $P(B) > 0$ such that (3.2) is violated for t on B . (This follows from the fact that both sides of the equality of (3.2) are right continuous.) Without loss of generality, we assume that on B :

$$S_j(t) > \frac{E \left(\int_t^1 x(s) dD_j(s) \mid \mathcal{F}_t \right)}{x(t)}.$$

We define

$$\begin{aligned}\theta_k(\omega, s) &= 0 \quad \forall \omega \in \Omega, k = 1, 2, \dots, j-1, j+1, \dots, J, \\ \theta_j(\omega, s) &= -1_{B \times (t, 1]}(\omega, s) \\ v(\omega, s) &= 1_{B \times [t, 1]}(\omega, s) (S_j(\omega, t) - D_j(\omega, s) + D_j(\omega, t)).\end{aligned}$$

As defined, θ is bounded and predictable, the latter of which follows from the fact that θ is left continuous; and v is an integrable variation process.

We claim that θ finances v . The fact that (θ, v) satisfy (2.3) and (2.4) for $s \in [0, 1]$ and on $\Omega \setminus B$ and for $s \in [0, t]$ on B is obvious. On B for $s \in (t, 1]$, we have

$$\begin{aligned}&\theta(0)^\top S(0) + \int_0^s \theta(u)^\top dS(u) + \int_0^{s-} \theta(u)^\top dD(u) - v(s-) \\ &= S_j(t) - S_j(s) + D_j(t) - D_j(s-) - v(s-) \\ &= S_j(t) - S_j(s) + D_j(t) - D_j(s-) - S_j(t) + D_j(s-) - D_j(t) \\ &= -S_j(s) = \theta_j(s) S_j(s) = \theta(s)^\top S(s).\end{aligned}$$

It is also clear that $\theta(1)^\top \Delta D(1) = \Delta v(1)$. Thus v is financed by θ and v is marketed. The price of v at time zero is zero. But we note the following

$$\begin{aligned}
\psi(v) &= E \left(\int_{[0,1]} x(s) dv(s) \right) \\
&= E \left(x(t) S_j(t) - \int_t^1 x(s) dD_j(s) \right) \\
&> E \left(\frac{x(t) E \left(\int_t^1 x(s) dD_j(s) \mid \mathcal{F}_t \right)}{x(t)} - \int_t^1 x(s) dD_j(s) \right) \\
&= 0,
\end{aligned}$$

a contradiction. ■

Appendix III: Proof for Proposition 4.2.

Proof. Let $\theta \in \Theta[\beta, S]$ finances $v \in V$. We want to show that v is financed by θ given (β^*, S^*) . It is clear that $\theta(1)^\top \Delta D(1) = \Delta v(1)$ a.s.. We are left to verify that

$$\begin{aligned}
\theta(t)^\top S^*(t) &= \theta(0)^\top S^*(0) + \int_0^t \theta(s)^\top dS^*(s) \\
&\quad + \int_0^{t-} \beta^*(s) \theta(s)^\top dD(s) - \int_0^{t-} \beta^*(s) dv(s) \quad \forall t \in [0, 1] \quad a.s. \quad (A.1)
\end{aligned}$$

Now define

$$G(t) = \theta(t)^\top \Delta D(t) - \Delta v(t)$$

and

$$G^*(t) = \beta^*(t) G(t).$$

The process

$$\theta(t)^\top S(t) + G(t) = \theta(0)^\top S(0) + \int_0^t \theta(s)^\top dS(s) + \int_0^t \theta(s)^\top dD(s) - v(t) \quad \forall t \in [0, 1] \quad a.s. \quad (A.2)$$

is a semimartingale, since the first integral on the right-hand-side is a semimartingale (cf. Theorem IV.20 of Meyer (1976)), and the second and the third terms are processes of bounded variation. Here we remark that $\{\theta(t)^\top S(t)\}$ may not be a semimartingale since if D or v is not continuous, then it is not right continuous, which is a defining property of a semimartingale. Now using the differentiation rule for semimartingales (cf. Corollary IV.23 of Meyer (1976)) we have

$$\begin{aligned}
d(\theta(t)^\top S^*(t) + G^*(t)) &= d\left(\beta^*(t) \left(\theta(t)^\top S(t) + G(t)\right)\right) \\
&= \beta^*(t-)d\left(\theta(t)^\top S(t) + G(t)\right) + \left(\theta(t-)^\top S(t-) + G(t-)\right)d\beta^*(t) + d[\beta^*, \theta^\top S + G]_t, \quad (\text{A.3})
\end{aligned}$$

where $[\cdot, \cdot]$ denotes the joint variation process (cf. Jacod (1979, 2.31)). The second term on the right-hand-side of (A.3) can be rewritten as:

$$\begin{aligned}
\left(\theta(t-)^\top S(t-) + G(t-)\right)d\beta^*(t) &= \left(\theta(t)^\top S(t) + G(t) - \theta(t)^\top \Delta S(t) - \theta(t)^\top \Delta D(t) + \Delta v(t)\right)d\beta^*(t) \\
&= \theta(t)^\top S(t-)d\beta^*(t), \quad (\text{A.4})
\end{aligned}$$

where the first equality follows from (A.2) and the second equality follows from the definition of $G(t)$.

We can rewrite the third term on the right-hand-side of (A.3) as:

$$d[\beta^*, \theta^\top S + G]_t = \theta(t)^\top d[\beta^*, S]_t + \theta(t)^\top d[\beta^*, D]_t - d[\beta^*, v]_t \quad (\text{A.5})$$

by Theorem IV.18 of Meyer (1976). Applying the differentiation rule for semimartingales again we get:

$$\begin{aligned}
dS^*(t) &= d(\beta^*(t)S(t)) \\
&= \beta^*(t-)dS(t) + S(t-)d\beta^*(t) + d[\beta^*, S]_t. \quad (\text{A.6})
\end{aligned}$$

Finally, we substitute (A.4) and (A.5) into (A.3) and use (A.6) to get:

$$d\left(\theta(t)^\top S^*(t) + G^*(t)\right) = \theta(t)^\top dS^*(t) + \beta^*(t)\theta(t)^\top dD(t) - \beta^*(t)dv(t), \quad (\text{A.7})$$

where we have used the fact that

$$\beta^*(t-)dD(t) + d[\beta^*, D]_t = \beta^*(t)dD(t)$$

and

$$\beta^*(t-)dv(t) + d[\beta^*, v]_t = \beta^*(t)dv(t);$$

cf. Corollary IV.23 of Meyer (1976). Equation (A.7) is just the differential form of (A.1). Thus θ finances v given (β^*, S^*) . This naturally implies that $\theta \in \Theta[\beta^*, S^*]$. The arguments for the converse are similar.

The proof for the rest of the assertion is easy. ■

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